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Discontinuous wave fronts propagation in anisotropic layered media

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Abstract

The problem of dynamic interaction of wave phase fronts with anisotropic elastic media interfaces is considered. A technique based on joint use of the ray theory, locally plane approach and theory of stereomechanical impact is elaborated. It is employed for the investigation of discontinuous waves propagation in anisotropic tectonic structures. The cases of interaction of quasi-longitudinal and quasi-shear discontinuous waves with the interfaces separating different anisotropic elastic media are treated. The issues are considered which are associated with the wave front surfaces bifurcations, generation of their singularities and caustics, as well as with stress concentration and formation of zones where the stresses tend to infinity.

Keywords: Anisotropic media; Discontinuous waves; Interface; Interaction; Ray method

1. Introduction

There is a large variety of problems of propagation of waves generated by impact action and characterized by discontinuities of strains and stresses at their front surfaces. Every so often in this cases, it is not necessary to study the field functions behaviour in the whole domain of a medium, but is expedient to consider the process of the wave motion as a signal, which can identify the mechanical system properties. This kind of problems is associated with the reconnaissance of mineral resources via the use of explosions, with the calculation of the phenomena of reflection–refraction of seismic waves in tectonic structures, as well as with the questions of the wave propagation and decay in layered composite materials.

In these cases, the problem of investigation of discontinuous waves propagation in elastic media is connected with the questions of geometrical construction of moving field functions discontinuities and calculation of their polarizations and magnitudes presenting the most comprehensive information on a

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wave front and intensity of the impulse carried by it at the every point of the front. The methods of geometrical optics, specifically, a zero approach of the ray method, providing an adequate quantitative description of a wide class of the wave phenomena of different physical nature, play an important role in the setting up and solution of the mentioned problems (Karal and Keller, 1959; Ogilvy, 1990; Petrashen, 1980; Podilchuk and Rubtsov, 1988, 1996).

The ray method allows to separate a function of the wave optical way length (or eikonal) and to construct a system of rays and fronts of a discontinuous wave with the help of the eikonal equation. This problem may be solved comparatively easy for isotropic media, but there again some difficulties arise when there is a necessity to investigate the interaction of a wave with the interface between the media with differing optical properties (lenses, inhomogeneities, etc.). In these events, the aggregates of the rays are produced, which have common envelopes (caustics), where the rays are focussed and the field intensity increases indefinitely. In geometrical optics, the caustics classification is performed on the basis of the theory of singularities of differentiable mappings—the theory of catastrophes (Arnold, 1974; Arnold et al., 1982; Kravtsov and Orlov, 1980; Poston and Stewart, 1978).

But the physical pictures of the dynamic phenomena are drastically complicated when discontinuous waves propagation in anisotropic elastic media is investigated, because of the field functions become vectorial; there are three kinds of waves for every ray direction, which are distinguished by their polarization; the waves phase velocities depend on both the waves polarization and direction of propagation; the ray velocities differ from the phase ones and there is not always one-to-one conformity between their directions. The phenomenon of the wave diffraction in the interfaces between elastic anisotropical media also is radically complicated, as the appropriate correlations by Snellius become essentially non-linear, because of the fact that the phase velocities of the reflected and refracted waves propagation cease to be known beforehand. For this reason it is necessary to solve systems of non-linear equations in order to determine the directions of the rays emanated from the boundary surfaces. The possible non-uniqueness of their solutions may cause the advent of caustics even at the incidence of a regular shock wave on a plane interface, which cannot be in homogeneous isotropical media, and generate more wide diversity of qualitatively different phenomena in the processes of reflection–refraction.

The problems of interaction of incident waves with the boundary surfaces interfacing anisotropical elastic media are normally solved through the construction of the refraction vector functions (Fedorov, 1968), which, in essence, represents a graphical method. Ogilvy (1990) applied a similar approach to investigation of a mirage phenomena in anisotropic heterogeneous elastic media. In the present paper, the method of continuation by a parameter jointly with the Newton method (Gulyayev et al., 1982) is used, which permits to identify bifurcational states of wave front transformation with the best efficiency.

2. Constitutive equations

Let the motion of a homogeneous anisotropical elastic medium characterized by the elasticity parameters $C_{ki,pq} = \text{const}$ and density $\rho = \text{const}$ be described by the equations

$$\sum_{k,p,q=1}^{3} \lambda_{ik,pq} \frac{\partial^2 u_q}{\partial x_k \partial x_p} - \frac{\partial^2 u_i}{\partial t^2} = 0 \quad (i = 1, 2, 3), \tag{1}$$

where $\lambda_{ik,pq} = C_{ik,pq}/\rho$; x_1, x_2, x_3 is the Cartesian coordinate system; u_1, u_2, u_3 the components of the elastic displacement vector.

Consider the system (1) solution in the form of a plane monochromatic wave with the wave number k and the phase velocity υ. Its fronts are the surfaces possessing the constant phases

$$\mathbf{n} \cdot \mathbf{r} - \upsilon t = \text{const}, \tag{2}$$

which coincide locally with the areas perpendicular to the unit vector $\mathbf{n}$ and moving with the velocity $\boldsymbol{\upsilon} = \upsilon \cdot \mathbf{n}$.

The question of the wave polarization vector $\mathbf{A}$ and phase velocity $\boldsymbol{\upsilon}$ determination for the selected direction $\mathbf{n}$ is solved on the basis of the system of homogeneous algebraic equations relative A_i (Fedorov, 1968; Gulyayev et al., 1997; Petrashen, 1980)

$$\sum_{k,p,q=1}^{3} \lambda_{ik,pq} n_k n_p A_q - \upsilon^2 A_i = 0 \quad (i = 1, 2, 3). \tag{3}$$

Its matrix

$$\Lambda_{iq} = \sum_{k,p=1}^{3} \lambda_{ik,pq} n_k n_p \quad (i, q = 1, 2, 3). \tag{4}$$

possesses the properties of symmetry and positive definiteness.

From condition of the system (3) non-trivial solution existence, the eigen-value problem stems

$$\left| \sum_{k,p=1}^{3} \lambda_{ik,pq} n_k n_p - \upsilon^2 \delta_{iq} \right| = 0. \tag{5}$$

It is possible to find three values of the velocity

$$\upsilon_1^2(\mathbf{n}) > \upsilon_2^2(\mathbf{n}) \geqslant \upsilon_3^2(\mathbf{n}) > 0 \tag{6}$$

with the help of Eq. (5) and three types $A^{(r)} (r = 1, 2, 3)$ of the wave polarization

$$\sum_{k,p,q=1}^{3} \lambda_{ik,pq} n_k n_p A_q^{(r)} - \upsilon_r^2 A_i^{(r)} = 0 \quad (i = 1, 2, 3), \tag{7}$$

for each direction $\mathbf{n}$. The waves are quasi-Primary (qP) and quasi-Secondary (qS).

These polarization vectors satisfy the orthogonality condition

$$\mathbf{A}^{(i)}(\mathbf{n}) \cdot \mathbf{A}^{(k)}(\mathbf{n}) = \delta_{ik} \quad (i = 1, 2, 3) \tag{8}$$

for every selected direction $\mathbf{n}$.

If a discontinuous wave is considered, equality (2) provides, that its front surface may be represented by the correlation

$$\tau(x_1, x_2, x_3) - t = 0, \tag{9}$$

where the function $\tau(x_1, x_2, x_3)$ has to satisfy the first order partial differential equation (Fedorov, 1968; Petrashen, 1980)

$$\sum_{i,k,p,q=1}^{3} \lambda_{ik,pq} p_k p_p A_q^{(r)} A_i^{(r)} = 1. \tag{10}$$

It generalizes the eikonal equation of geometrical optics to the case of anisotropic elastic waves.

The quantities $p_k (k = 1, 2, 3)$ included into (10) represent the components of the refraction vector $p_k \equiv \partial\tau/\partial x_k = n_k/\upsilon_r(\mathbf{n}) (k = 1, 2, 3)$.

For the wave front (9) to be constructed, Eq. (10) solution should be found. The equation is transformed to the system of ordinary differential equations

$$\partial x_k/\partial\tau = \sum_{i,p,q=1}^{3} \lambda_{ik,pq} p_p A_q^{(r)} A_i^{(r)}, \qquad \partial p_k/\partial\tau = 0 \quad (k = 1,2,3) \tag{11}$$

with the help of the method of characteristics.

The first group of these equations describes the wave propagation along rays with the ray velocity $\boldsymbol{\xi} = \boldsymbol{\xi}^{(r)}(\mathbf{n}, x_k)$. The rays are rectilinear and, in the general case, not orthogonal to the wave front surface.

The built up system of rays and fronts allows to proceed to the determination of the wave intensity in the vicinity of its front. For the realization to be performed, it is convenient to use Eq. (1) solution expansion in series along a ray as follows:

$$u_q = \sum_{m=0}^{\infty} u_q^{(m)}(x_1,x_2,x_3) f_m[t - \tau(x_1,x_2,x_3)] \quad (q = 1,2,3), \tag{12}$$

where the functions f_m, satisfying the correlations $f'_m(y) = f_{m-1}(y)$, are supposed to possess discontinuities of their derivatives, for example, of the order $n+2$ (Petrashen, 1980).

If the problem of investigation of the wave behaviour in the front nearest neighbourhood is set up, only one term $m = 0$ is retained in (12) and for the vector $\mathbf{u}^{(0)}$ to be calculated, the system of homogeneous equations

$$\sum_{k,p,q}^{3} \lambda_{ik,pq} p_k p_p u_q^{(0)} - u_i^{(0)} = 0 \quad (i = 1,2,3) \tag{13}$$

is used. Its solution is represented in the form (Petrashen, 1980)

$$u_q^{(0)} = \frac{c_0(\alpha,\beta) \cdot A_q^{(r)}(\alpha,\beta,\tau)}{\sqrt{J(\alpha,\beta,\tau)}} \quad (q = 1,2,3), \tag{14}$$

where α, β, τ is the system of ray coordinates and the functional determinant $J = \partial(x_1,x_2,x_3)/\partial(\alpha,\beta,\tau)$ of the transformation of the ray coordinates into Cartesian ones is the measure of the ray divergence in the ray tube.

The presented correlations permit to trance the evolution of a discontinuous wave front and to calculate magnitudes of the field functions discontinuities on its surface outside the interface between anisotropic elastic media with differing properties.

3. Kinematics of interaction with interface

Consider the interaction of an axisymmetrical impulse wave emitted from a spherical source C with a plane interface G between two transversally isotropic elastic media I and II. Let their axes of symmetry are parallel to each other and perpendicular to the plane G. Assume "locally plane" approach, according to which at the spot of the initial shock wave incidence on the separating plane G, all the reflected and refracted waves are situated in the same plane, i.e. the third components of their polarization vectors equal zero. This assumption permits to introduce the angles θ_ν and $\bar{\theta}_\mu$, with which the reflected and refracted waves move away from the boundary G and to use the Snellius generalized law expressed by the equalities

$$\frac{\sin\theta}{\upsilon(\theta)} = \frac{\sin\theta_\nu}{\upsilon_\nu(\theta_\nu)} = \frac{\sin\bar{\theta}_\mu}{\bar{\upsilon}_\mu(\bar{\theta}_\mu)} \quad (\nu,\mu = 1,2). \tag{15}$$

Here θ is the incidence angle of the wave in the medium I; θ_ν ($\nu = 1,2$) the angles between the reflected rays and the axis Ox_2; $\bar{\theta}_\mu$ ($\mu = 1,2$) the angles between the rays penetrated into the medium II and the axis Ox_2.

The indexes value $\nu = \mu = 1$ corresponds to the quasi-longitudinal (qP) waves, the value $\nu = \mu = 2$ is associated with the quasi-shear (qS) waves.

The difference between correlation (15) and usual form of the Shellius law is caused by the denominators $\upsilon_\nu(\theta_\mu)$, $\bar{\upsilon}_\mu(\theta_\mu)$ dependence on the unknown angles θ_ν, $\bar{\theta}_\mu$ and, implicitly, on the incidence angle θ. So, in order to find the values of the angles θ_ν, $\bar{\theta}_\mu$ $(\nu, \mu = 1, 2)$ corresponding to the set θ, it is necessary to solve non-linear equations system (15). With this aim in view, the Newton method jointly with the method of continuation by a parameter are used (Gulyayev et al., 1982). Consider the first equality of system (15). Let the angles $\theta_\nu^{(n)}$, corresponding to some one value $\theta = \theta^{(n)}$ are determined. Prescribe a small increment $\Delta\theta^{(n)}$ to the quantity $\theta^{(n)}$ and find the appropriate increment $\Delta\theta_\nu^{(n)}$. For this purpose take the first equation of system (15) in the form

$$\sin(\theta^{(n)} + \Delta\theta^{(n)}) \cdot \upsilon_\nu(\theta_\nu^{(n)} + \Delta\theta_\nu^{(n)}) - \sin(\theta_\nu^{(n)} + \Delta\theta_\nu^{(n)}) \cdot \upsilon(\theta^{(n)} + \Delta\theta^{(n)}) = 0. \tag{16}$$

Separating its linear part, one gains

$$\Delta\theta_\nu^{(n)} \approx [\sin\theta_\nu^{(n)} \cdot \partial\upsilon(\theta^{(n)})/\partial\theta - \cos\theta^{(n)} \cdot \upsilon_\nu(\theta_\nu^{(n)})] \times [\sin\theta^{(n)} \cdot \partial\upsilon_\nu(\theta_\nu^{(n)})/\partial\theta_\nu - \cos\theta_\nu^{(n)} \cdot \upsilon(\theta^{(n)})]^{-1}\Delta\theta^{(n)}. \tag{17}$$

After calculation $\Delta\theta_\nu^{(n)}$, the kinematic parameters $\theta^{(n+1)} = \theta^{(n)} + \Delta\theta^{(n)}$, $\theta_\nu^{(n+1)} = \theta_\nu^{(n)} + \Delta\theta_\nu^{(n)}$, $\upsilon(\theta^{(n+1)})$, $\upsilon_\nu(\theta_\nu^{(n+1)})$ are found. Linearizing the first equation of system (15) in the vicinity of the state $\theta^{(n+1)}$, $\upsilon(\theta^{(n+1)})$, $\theta_\nu^{(n+1)}$, $\upsilon_\nu(\theta_\nu^{(n+1)})$, the increments $\Delta\theta^{(n+1)}$, $\Delta\theta_\nu^{(n+1)}$ can be determined and so on.

In doing so it is necessary to take into account that inasmuch as relation (17) is approximate a mistake of the $\Delta\theta_\nu^{(n)}$ calculation grows with the n increase. For this reason in order to compensate the inaccuracy the appropriate residue of Eq. (15) should be added with the opposite sign to the right member of equality (17), as it is done in Newton's method. As result the computational scheme is deduced

$$\begin{aligned}\Delta\theta_\nu^{(n)} = {} & [\sin\theta_\nu^{(n)}\partial\upsilon(\theta^{(n)})/\partial\theta - \cos\theta^{(n)}\upsilon_\nu(\theta_\nu^{(n)})] \cdot [\sin\theta^{(n)}\partial\upsilon_\nu(\theta_\nu^{(n)})/\partial\theta_\nu - \cos\theta_\nu^{(n)} \cdot \upsilon(\theta^{(n)})]^{-1}\Delta\theta^{(n)} \\ & - \sin\theta^{(n)} \cdot \upsilon_\nu(\theta_\nu^{(n)}) + \sin\theta_\nu^{(n)} \cdot \upsilon(\theta^{(n)})\end{aligned} \tag{18}$$

combining the method of continuation by a parameter and Newton's method (Gulyayev et al., 1982). Its accuracy increases with the $\Delta\theta^{(n)}$ decrease.

The derivative $\partial\upsilon/\partial\theta$ to be calculated one must differentiate the left member of Eq. (5) with respect to θ and equate it with zero. Then we have

$$\sum_{i=1}^{3}\sum_{q=1}^{3} \frac{\partial b_{iq}}{\partial\theta} \cdot b_{iq}^{*} = 0, \tag{19}$$

where

$$b_{iq} = \sum_{k,p=1}^{3} \lambda_{ik,pq} n_k n_p - \upsilon^2\delta_{iq}, \tag{20}$$

b_{iq}^{*} is the corresponding algebraic adjunct.

Substituting

$$\partial b_{iq}/\partial\theta = \sum_{k,p=1}^{3} \lambda_{ik,pq}(n_k\partial n_p/\partial\theta + n_p\partial n_k/\partial\theta) - 2\upsilon\partial\upsilon/\partial\theta \cdot \delta_{iq} \tag{21}$$

into (19) we gain the equation determining $\partial\upsilon/\partial\theta$.

In the same manner the derivatives $\partial \upsilon_v(\theta_v^{(n)})/\partial\theta_v$ are determined.

The performance of calculation with the use of scheme (18) is possible when some initial state $\theta^{(0)}$, $\upsilon(\theta^{(0)})$, $\theta_v^{(0)}$, $\upsilon_v(\theta_v^{(0)})$ is preset. It is convenient for the considered case to select $\theta^{(0)} = 0$, $\upsilon(0)$, $\theta_v^{(0)} = 0$, $\upsilon_v(0)$.

When the denominator of correlation (18) is not equal to zero, it allows to find the single increment $\Delta\theta_v$ corresponding to the incidence angle θ. So the equality

$$\sin\theta^{(n)} \cdot \partial\upsilon_v(\theta_v^{(n)})/\partial\theta_v - \cos\theta_v^{(n)} \cdot \upsilon(\theta^{(n)}) = 0 \tag{22}$$

is the condition of the solution bifurcation (Vainberg and Trenogin, 1969). With the purpose to continue the solution through this state, it is necessary to add the terms of second order into (18).

The condition of possible non-uniqueness of the system (17) solutions derives from the phenomenon of the rays convergency (tangency) and intersection of the reflected and refracted rays after the interaction of the incident rays with the interface G and the manifold of similar critical states is attended with formation of the rays family envelopes (caustics).

Inasmuch as the wave front singularities are generated in the caustics, its ray focussing occurs in some spots which is accompanied by the functional determinant J transformation into zero and by the infinite increase of the field functions intensities in the places of the geometrical singularities. In the caustics, the wave phase also changes to the opposite one (Kravtsov and Orlov, 1980).

4. Discussion of results

At first consider the peculiarities of a discontinuous wave propagation in an unbounded transversally isotropic medium. Let a normal uniform pressure is instantly applied to the surface of a spherical cavity C of radius $R = 1$. It initiates not only a qP discontinuous wave, as it occurs in an isotropic medium, but also qS discontinuous waves whose front surfaces possess axial symmetry. Thanks to it intensity of the qS-wave ($r = 3$) polarized orthogonally to the first ones ($r = 1, 2$) is equal to zero and hereafter it will not be investigated. The problem is to construct the evolving surfaces of these waves fronts and to investigate their bifurcations.

At the case under study the tensor of elastic constants is represented by the matrix

$$(C_{\alpha\beta}) = \begin{Vmatrix} L & 0 \\ 0 & M \end{Vmatrix}; \quad L = \begin{Vmatrix} \lambda + 2\mu & \lambda - l & \lambda \\ \lambda - l & \lambda + 2\mu - p & \lambda - l \\ \lambda & \lambda - l & \lambda + 2\mu \end{Vmatrix},$$

$$M = \mathrm{diag}(\mu - m, \mu, \mu - m).$$

Thus, the three parameters l, m, p characterize the considered media difference from an isotropic medium with the Lame parameters λ, μ.

Let the axis Ox_2 coincides with the axis of symmetry of the medium elastic properties. Consider the modes of sections of these waves fronts ($r = 1, 2$) by the plane $x_3 = 0$ depending on correlations between the anisotropy parameters l, m, p with the parameters λ, μ and ρ being fixed.

With the target of systematization and classification of basic types of the discontinuous qP- and qS-waves front surfaces depending on the parameters l, m, p values their magnitudes are selected which satisfy the inequalities (Petrashen, 1980)

$$\begin{aligned} &(\lambda + \mu - m - l)^2 > (\lambda + 2\mu)(\lambda + \mu + m - p), \\ &(\lambda + \mu - m - l)^2 > (\lambda + 2\mu - p)(\lambda + \mu + m), \\ &(\lambda + \mu - m - l)^2 < (\lambda + \mu + m - p)(\lambda + \mu + m) \end{aligned} \tag{23}$$

representing the conditions of origination of three types of singularities.

Besides, the elasticity parameters are bound to satisfy the relationship

$$(\lambda + \mu - l)^2 > (\lambda + 2\mu - p)(\lambda + \mu)$$

ensuring positiveness of the medium internal energy.

To cover all the possible types of the discontinuous wave fronts, emanating from the spherical source, the values of the parameters l, m, p were varied and preset as fractions of λ, μ, $\lambda + 2\mu$ with the help of the multipliers a, b, c. The values of the parameters ρ, λ, μ, a, b, c as well as the types of singularities and numbers of the pictures in Fig. 1 illustrating axial sections of the appropriate fronts, are presented in Table 1. The waves fronts, shown in Fig. 1, were built as loci of the ray velocity vectors $\xi^{(2)}(\mathbf{n})$ points at fixed time

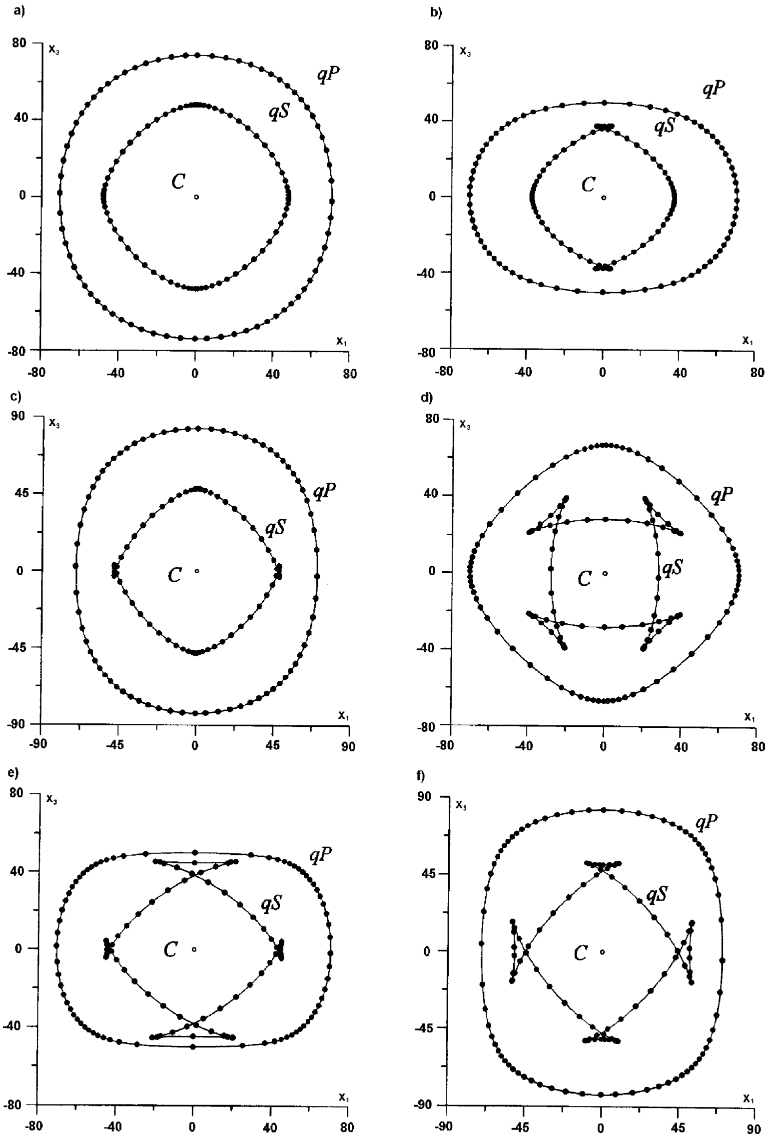

Fig. 1. Outlines of the qP- and qS-waves fronts.

Table 1
Elastic media properties

Case number	Singularity type	Figure	λ, μ, ρ	a [$l = a\lambda$]	b [$m = b\mu$]	c [$p = c(\lambda + 2\mu)$]
1	Without singularity	a		−0.5	0.3	−0.1
2	1	b		0.1	0.3	0.5
3	2	c	$\lambda = 2.502 \times 10^{10}$ Pa, $\mu = 1.965 \times 10^{10}$ Pa,	−0.5	−0.3	−0.4
4	3	d	$\rho = 3 \times 10^3$ kg/m^3	0.5	0.3	0.1
5	1 and 2	e		−0.3	0.3	0.5
6	1 and 2	f		−0.7	−0.7	−0.4

moments. The points of intersection of the fronts with the rays emanating from the cavity spherical surface with the angle increment 5° are identified by the black circlets.

Analysis of the gained results allows to draw inferences consisted with the conclusions of Petrashen (1980). The qP-wave front curves are always convex and enclose the curves of the qS-waves fronts. The following cases may occur depending on satisfaction of conditions (23).

Case 1. None of inequalities (23) is met, the singularities are absent, the both waves fronts make up smooth surfaces (Fig. 1a).

Case 2. The first inequality (23) is fulfilled. The singularity of the first type comes into being and in the axial section of the qS-wave front surface the bifurcation of "swallow tail" type is produced in the vicinity of the axis Ox_2. In the pointings, the values $J(\alpha, \beta, \tau)$ equal zero and the field functions $u_q^{(0)}$ in (14) tend to infinity.

Case 3. The second condition (23) is satisfied, the second type singularity takes place. It is characterized by the pS-wave front bifurcation and caustics generation in the neighbourhood of the axis Ox_1 (Fig. 1c).

Case 4. The third correlation (23) is true. The singularity of the third type manifests itself in generation of bifurcations and caustics in the vicinity of the rays inclined to the axes Ox_1 and Ox_2 (Fig. 1d).

Cases 5 and 6. The inequalities considered in cases 2 and 3 are met simultaneously. It is conceivable that the singularities of the first and second types occur concurrently (Fig. 1e and f). Trace the behaviour of the rays and normals to the fronts at regular and irregular points, displaying the rays and evolving fronts of the qP- and qS-waves separately for case 4 (Fig. 2a and b). All the parts of the qP-wave fronts (Fig. 2a) are regular because only one direction of the wave front normal **n** and one piece of the front correspond to every selected ray direction. All the qS-wave front segments, away from pointings, are regular as well. But in the vicinity of the pointings there are the ray directions, where three different directions of the normals **n** may correspond to a ray. At these points the wave front bifurcates, the functional determinant $J = \partial(x_1, x_2, x_3)/\partial(\alpha, \beta, \tau)$ degenerates and intensity of the wave function $\dot{u}_q$ tends to infinity (Vainberg and Trenogin, 1969). Condensation of rays in the points vicinity corroborates this conclusion.

The problem of dynamic interaction of the qP- and qS-wave fronts with the plane interface G between two transversally isotropical media was solved with the aid of the foregoing approach. The plane $x^2 = 1$ m is perpendicular to the axis Ox_2, coinciding with the symmetry axes of the media elasticity parameters. The first medium mechanical characteristics were determined by the parameters $\lambda_1 = 4.97 \times 10^{10}$ Pa, $\mu_1 = 3.91 \times 10^{10}$ Pa, $l_1 = -0.5\lambda_1$, $m_1 = -0.3\mu_1$, $p_1 = -0.4(\lambda_1 + 2\mu_1)$, $\rho_1 = 2.65 \times 10^3$ kg/m^3; for the second medium they constituted $\lambda_2 = 3.41 \times 10^{10}$ Pa, $\mu_2 = 1.36 \times 10^{10}$ Pa, $l_2 = -0.5\lambda_2$, $m_2 = -0.3\mu_2$, $p_2 = -0.5(\lambda_2 + 2\mu_2)$, $\rho_2 = 2.76 \times 10^3$ kg/m^3.

It was assumed that a normal uniform pressure was instantly applied to the surface of a spherical cavity of radius $R = 0.1$ m offset by 1 m from the interface. In spite of the fact that the pressure is uniform it induces the $qP_{(1-)}$-wave and the $qS_{(1-)}$-wave following the first one.

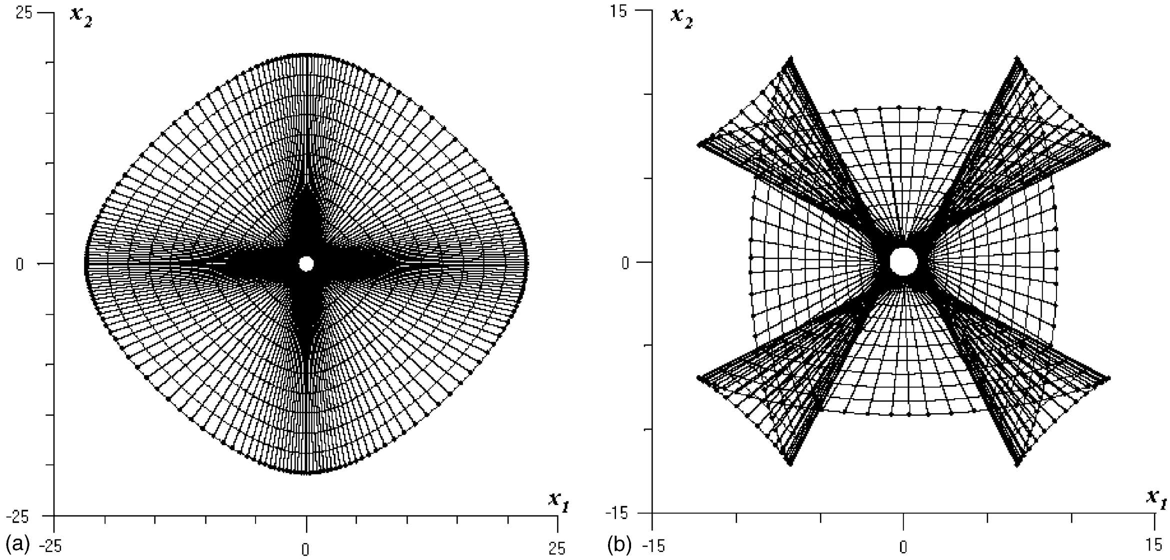

Fig. 2. Fronts of the $qP_{(1-)}$ (a) and $qS_{(1-)}$ (b) waves for the fourth combination of parameters.

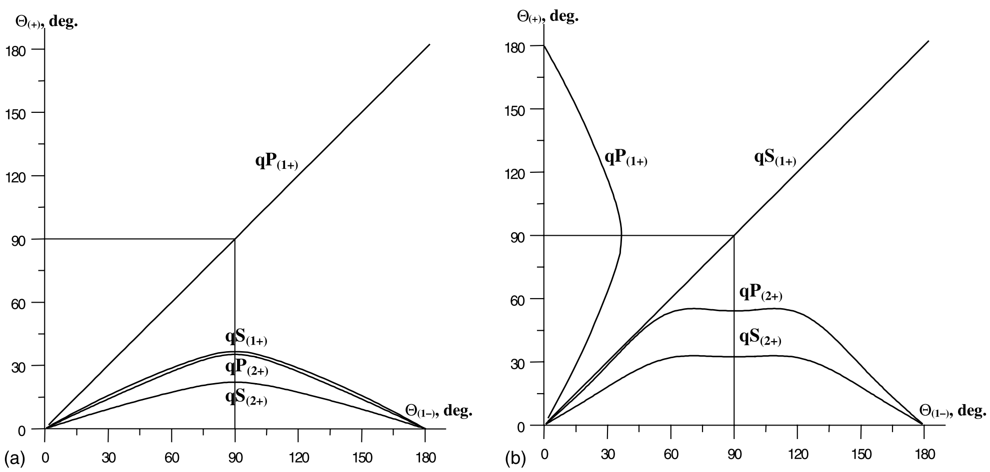

Fig. 3. Solutions of the Snellius equations for the $qP_{(1-)}$ (a) and $qS_{(1-)}$ (b) incident waves.

To determine directions of the rays reflected and refracted in the plane G, Snellius equation (15) were solved for both of the waves. Fig. 3 shows relationships between the angles of quasi-longitudinal (a) and quasi-shear (b) incident discontinuous waves (plotted as abscissas) and appropriate angles of the reflected and refracted waves (plotted as ordinates). At calculations it was assumed $\Delta\theta = \theta_{(1-)} = 0.2°$ in (16). Inasmuch as the functions of the angles of the waves $qP_{(1+)}$ (Fig. 3a) and $qS_{(1+)}$ (Fig. 3b) have the shapes of straight lines coming under angles 45° from the origins, one may conclude that at the selected orientation of the media properties symmetry axes the angles of reflection and incidence of the like waves have the same values. Here the number 1, 2 in the round brackets denote the media numbers, the signs "−" and "+" denote before and after interaction.

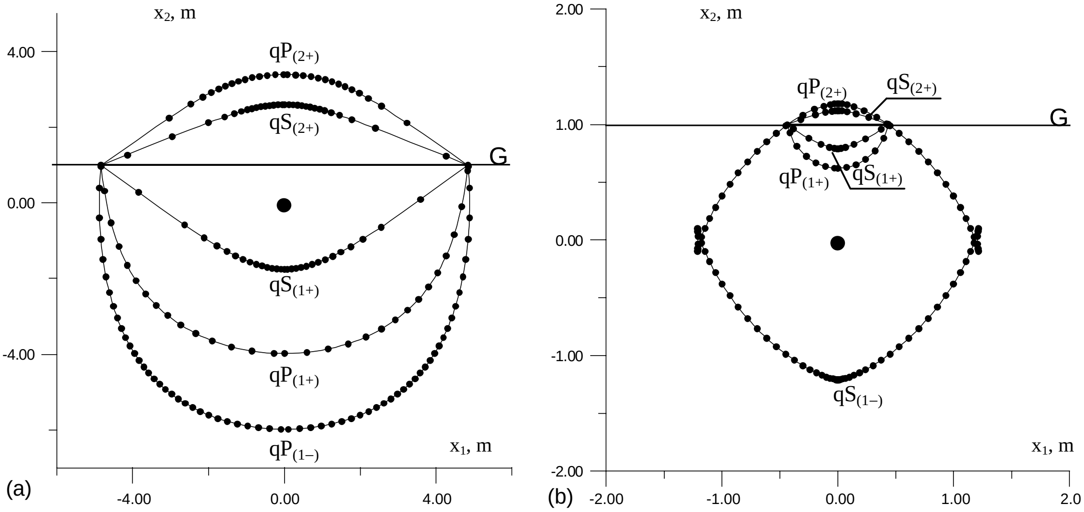

Fig. 4. Front outlines for the cases of the $qP_{(1-)}$ (a) and $qS_{(1-)}$ (b) incident waves.

The Snellius equation solutions for the quasi-longitudinal incident $qP_{(1-)}$-wave are unique (Fig. 3a), so critical states do not appear in these events. But for the second case, when the $qS_{(1-)}$-wave is incident, two values of the angle $\theta_{(1+)}$ for the reflected $qP_{(1+)}$-wave correspond to one value of the incidence angle $\theta_{(1-)}$ for $\theta_{(1-)} < 32.8°$ with the straight vertical line $\theta_{(1-)} = 32.8°$ being tangent to the curve $\theta_{(1+)}$ $(\theta_{(1-)})$ for the $qP_{(1+)}$ wave (Fig. 3b). This being so, it may be concluded that the angle value $\theta_{(1-)} = 32.8°$ is critical for the incident $qS_{(1-)}$-wave. In this state, condition (22) is satisfied and the values of the required variables approach infinity. The appropriate incident ray is limiting for the conventional ray theory applicability. With the incident angle exceeding, the Snellius equation loses its sense and the system of the rays and fronts becomes much more intricate. The conventional ray theory ceases its validity in these cases.

Shown in Fig. 4 are the sections of the incident, reflected and refracted waves fronts. Fig. 4a conforms to the $qP_{(1-)}$ incident wave and the incident angle value $\theta_{(1-)} = 85°$, Fig. 4b is appropriate to the $qS_{(1-)}$ incident wave and the angle value $\theta_{(1-)} = 32.8°$. Both the waves are generated by the same shock pressure applied to a small spherical cavity surface. There are not any front bifurcations for the chosen elasticity parameters of the media in the vicinity of the interface.

5. Conclusions

The problem of diffraction of discontinuous waves fronts at the interfaces between anisotropic elastic media is set up in the framework of a zero approximation of the ray theory. A software is elaborated with its use for the computer simulation of the waves fronts transformation and analysis of their field functions discontinuities magnitudes evolution.

The carried out calculations allowed to investigate phenomena of bifurcations of the wave front surfaces and trace the generation of reflected and refracted discontinuous waves in the interface between the media with different mechanical properties. It is established that three factors may be the reasons of the fronts bifurcations and caustics formation. Among these are the non-linear correlation between the directions of rays and a wave normal, the non-linear character of Snellius equations and non-linear outline of interfaces. In the vicinity of the wave front bifurcations, the field functions intensity tends to infinity.

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